Turbulent Rankine Vortices

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April 2008

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- Overview of key results in the theory of turbulence
- Motivation for a fresh perspective on turbulence
- The Rankine vortex
  - CFD simulation of 2D vortex decay
  - Analysis of fixed-radius 2D vortex decay
- The dynamic equilibrium of an ensemble of vortices
- Energy transfer and energy dissipation
- The Kolmogorov dissipation scale
- The Kolmogorov-Obukhov –5/3 law
  - And its high-wavenumber correction due to Pao
- Summary

Key results in the theory of turbulence

- Fluid equations of motion: Navier 1823, Stokes 1843
- Fluctuations around the mean: Reynolds 1895
- The *correlation function* is the fundamental variable for the description and analysis of turbulence: Taylor 1935
- ‘Frozen convection’ hypothesis: Taylor 1938
  - 2-point correlation function and 1-point time series related as FTs
- Fundamental equation for the propagation of the correlation function: von Kármán and Howarth 1938
- The ‘–5/3 power law’ energy spectrum of isotropic turbulence: Kolmogorov 1941, Obukhov 1941

Motivation for a fresh perspective (1)

- “Turbulence is the most important unsolved problem of classical physics.” (Richard Feynman)
- “I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.” (Horace Lamb)
- “Historically, there has been almost no interaction between the Kolmogorov theory of turbulence and the attempts to explain turbulence from the... Navier-Stokes equations… That is, the equations have not helped us understand Kolmogorov, and Kolmogorov has not helped solve the equations.” (Benoît Mandelbrot)

Motivation for a fresh perspective (2)

- Turbulence has a *mathematical* explanation…
  - The statistical theory of turbulence
- And it has a *physical* explanation…
  - Richardson (1922) 'eddy cascade' model: "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity…"
  - Batchelor (1953) vortex stretching mechanism
- But there is no obvious connection between the two…
  - Precisely how do the parameters of the statistical theory relate to the parameters of the eddy cascade model?
  - For that matter, what *are* the parameters of the eddy cascade model?
- And, as a result, turbulence theory appears incomplete.

Motivation for a fresh perspective (3)

- Turbulence is measured by finding the spectrum of the correlation function
  - The correlation function is of interest only when it measures something more than noise, *i.e*. when it measures the presence of coherent structures
  - By ‘coherent structures’ is meant *vortices*, which are a characteristic feature of turbulence
- Can we describe turbulence in terms of the dynamics of an ensemble of vortices?
  - For that matter, can we describe the dynamics of a single vortex?
  - Simplest model of a 2D vortex: The Rankine vortex

J. Macquorn Rankine (1820-1872)

The Rankine vortex

Fig. 139

Fig. 140

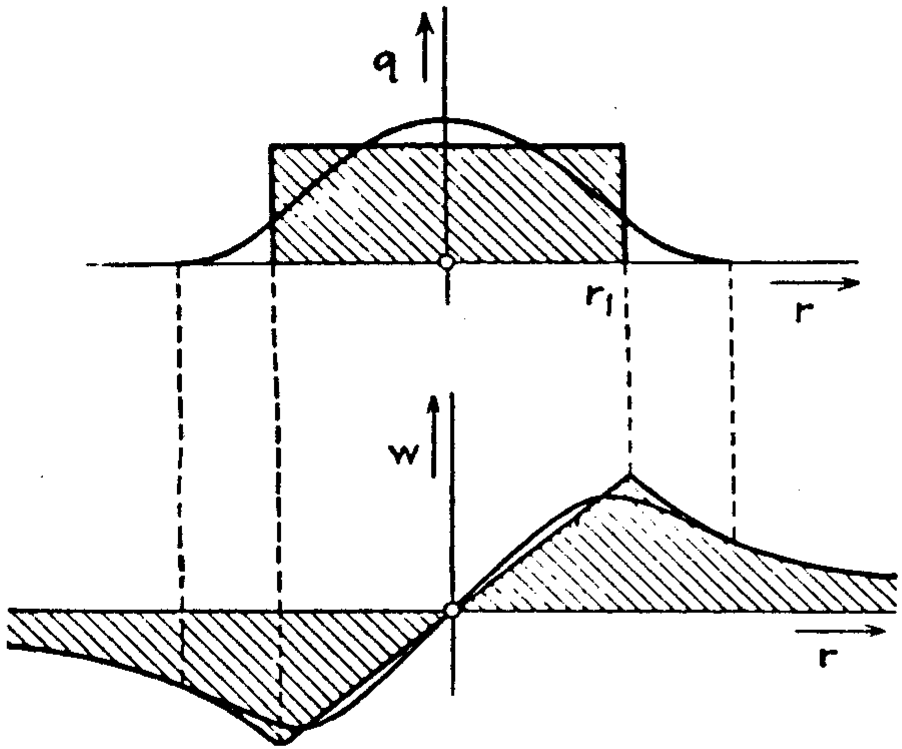


Fig. 139.—Simplified construction of a vortex filament by assuming a vortex core of constant rotation.

Fig. 140.—Distribution of velocity due to a vortex filament and that due to a vortex core surrounded by potential flow.

L Prandtl and O G Tietjens, *Fundamentals of Hydro- and Aeromechanics*, Dover (New York, 1934)

CFD simulation of 2D vortex decay

- Simple CFD model of a vortex in two dimensions
  - 2D axisymmetric flow
  - Initiated with ω = constant for $r \leq R$, $\omega = 0$ for $r > R$
  - Boundary condition $\boldsymbol{u} = 0$ at $r = 10R$
  - No external forces, flow allowed to evolve in time
- Observations
  - Steady decay with a core radius which is approximately constant
  - Eventually the vortex loses coherence as the core radius increases and the vortex expands to fill the entire region
  - Dynamical similarity with respect to changes in radius, velocity, and viscosity

Figure 1: Base case

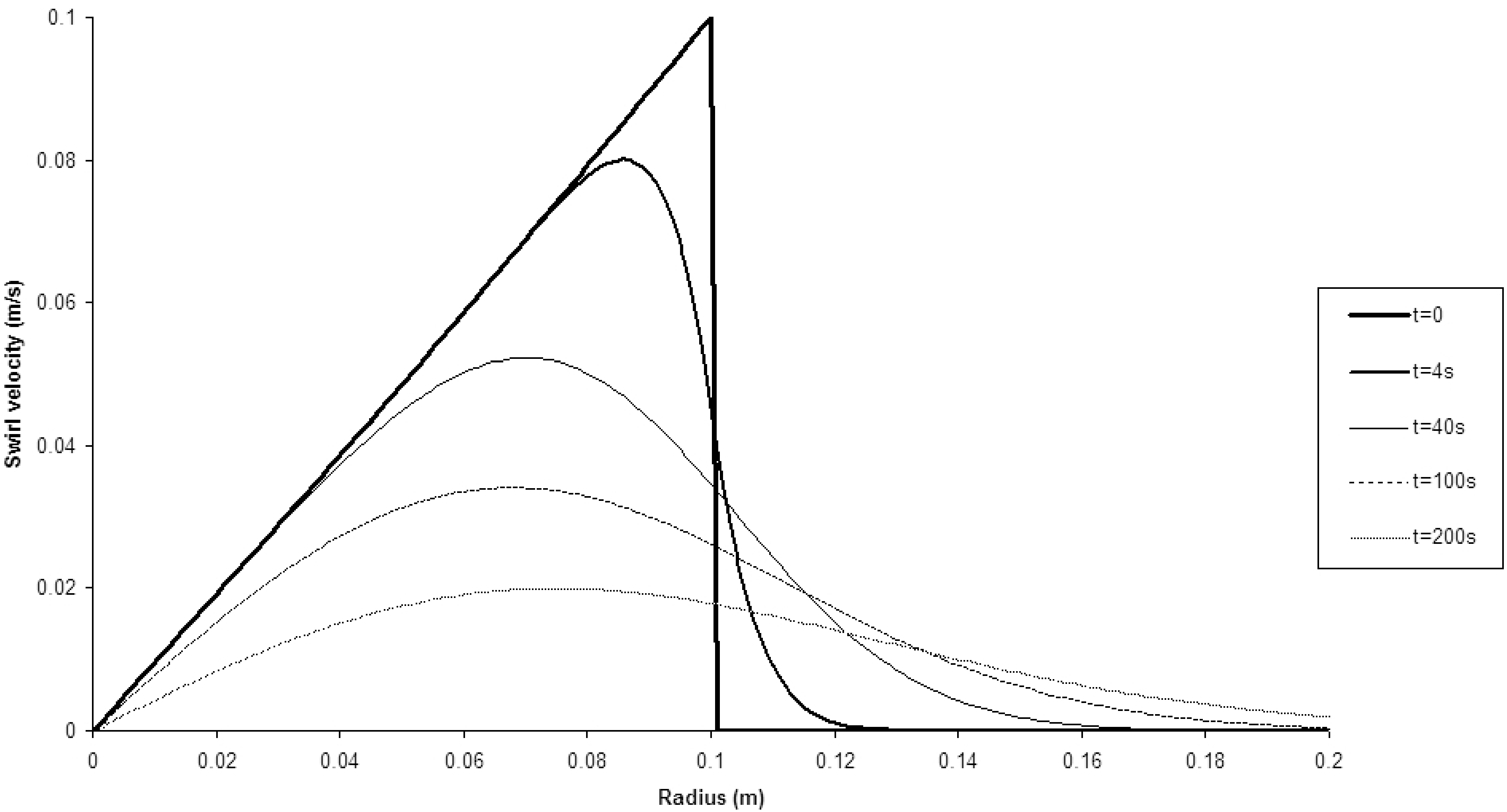


Figure 2: Radius increased by factor 10

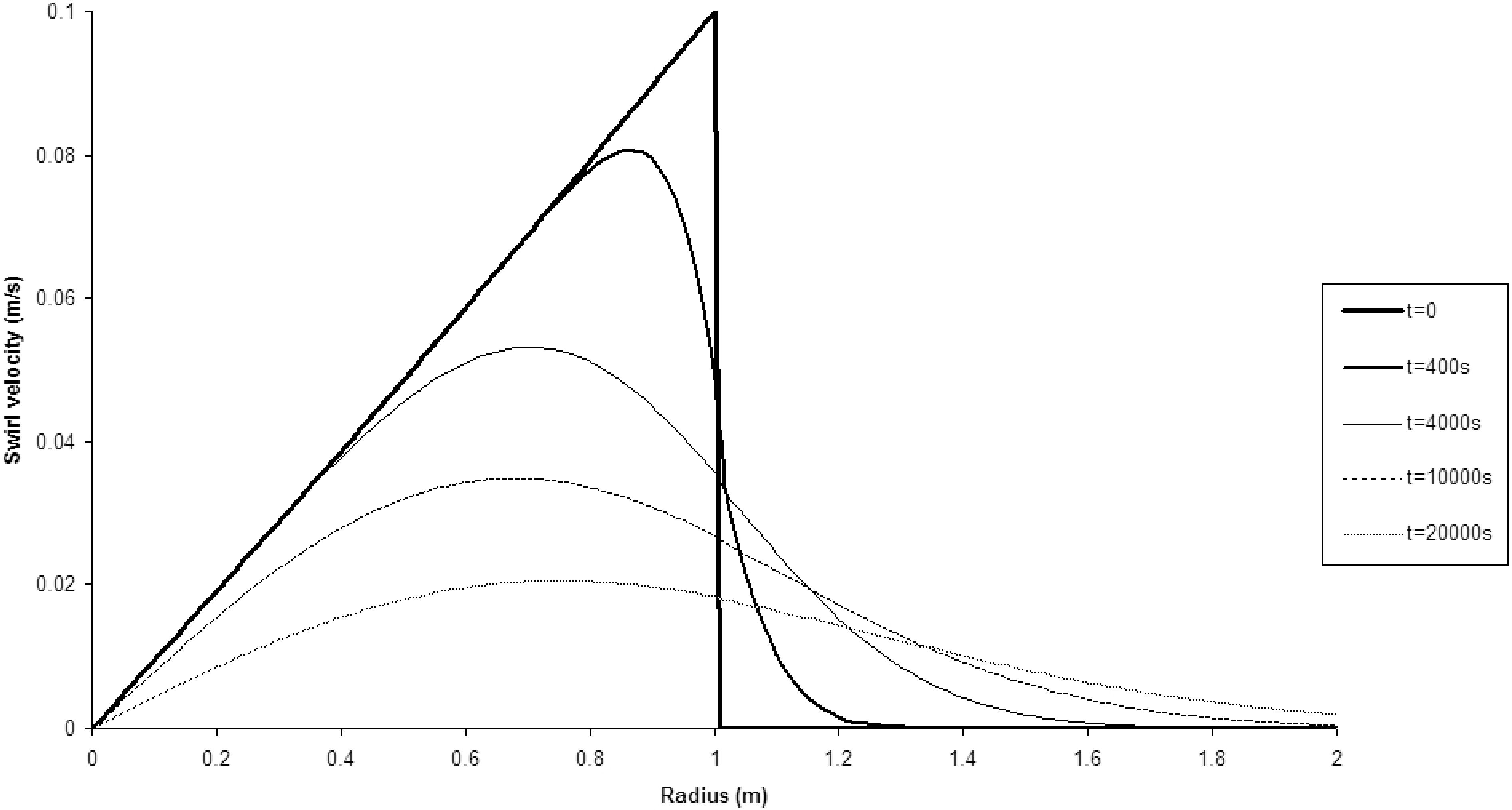


Figure 3: Swirl velocity increased by factor 10

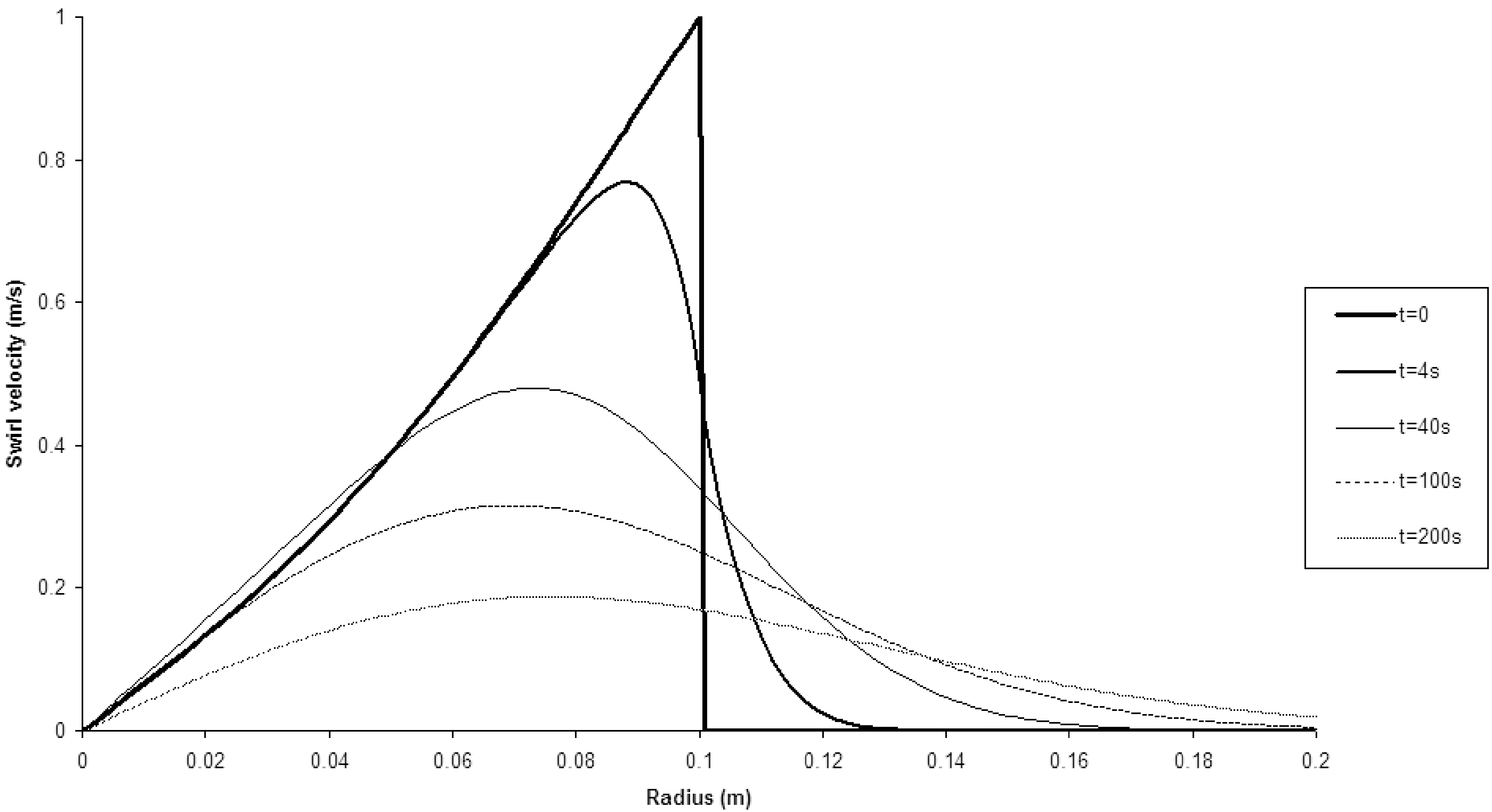
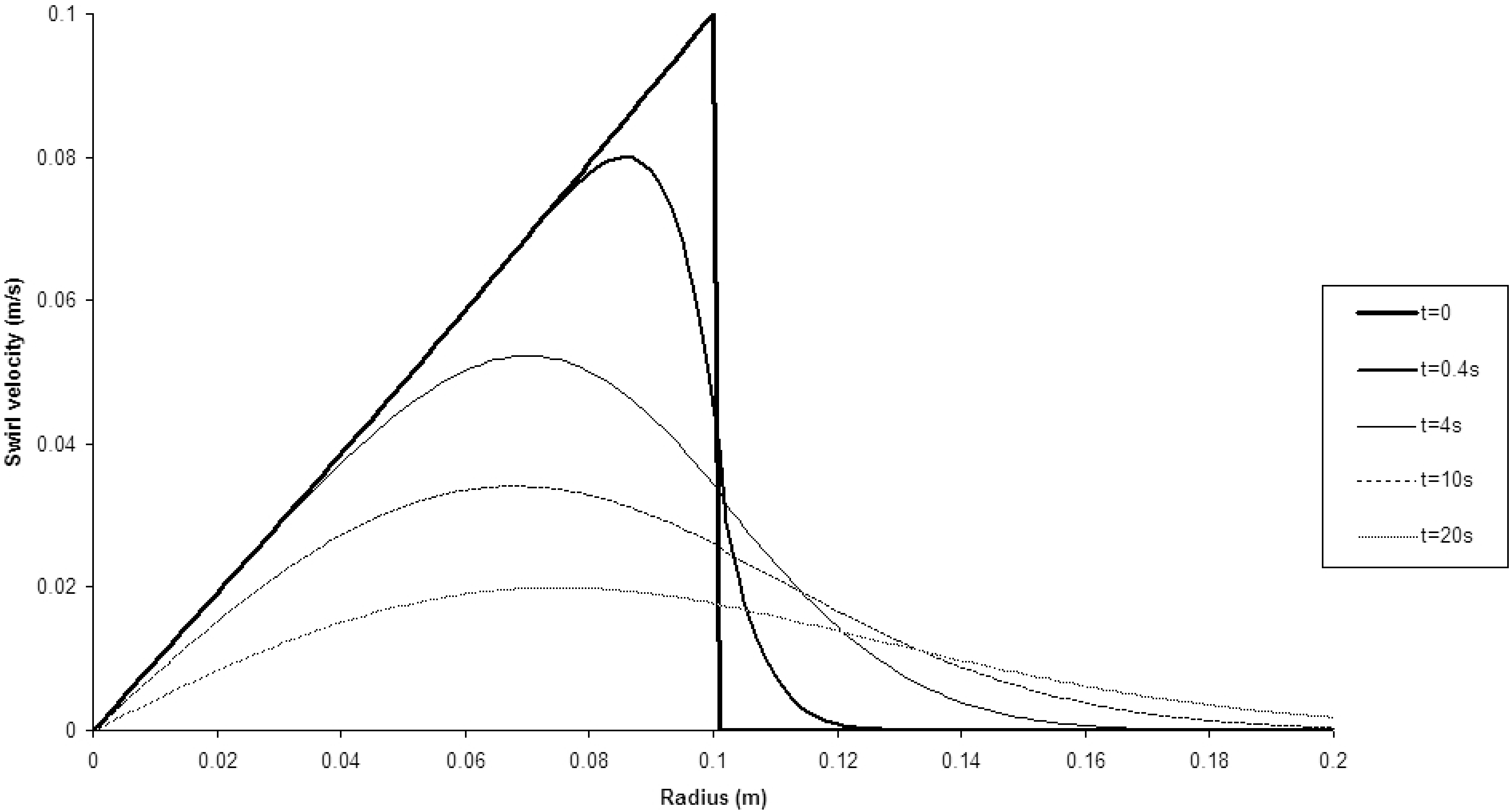


Figure 4: Viscosity increased by factor 10



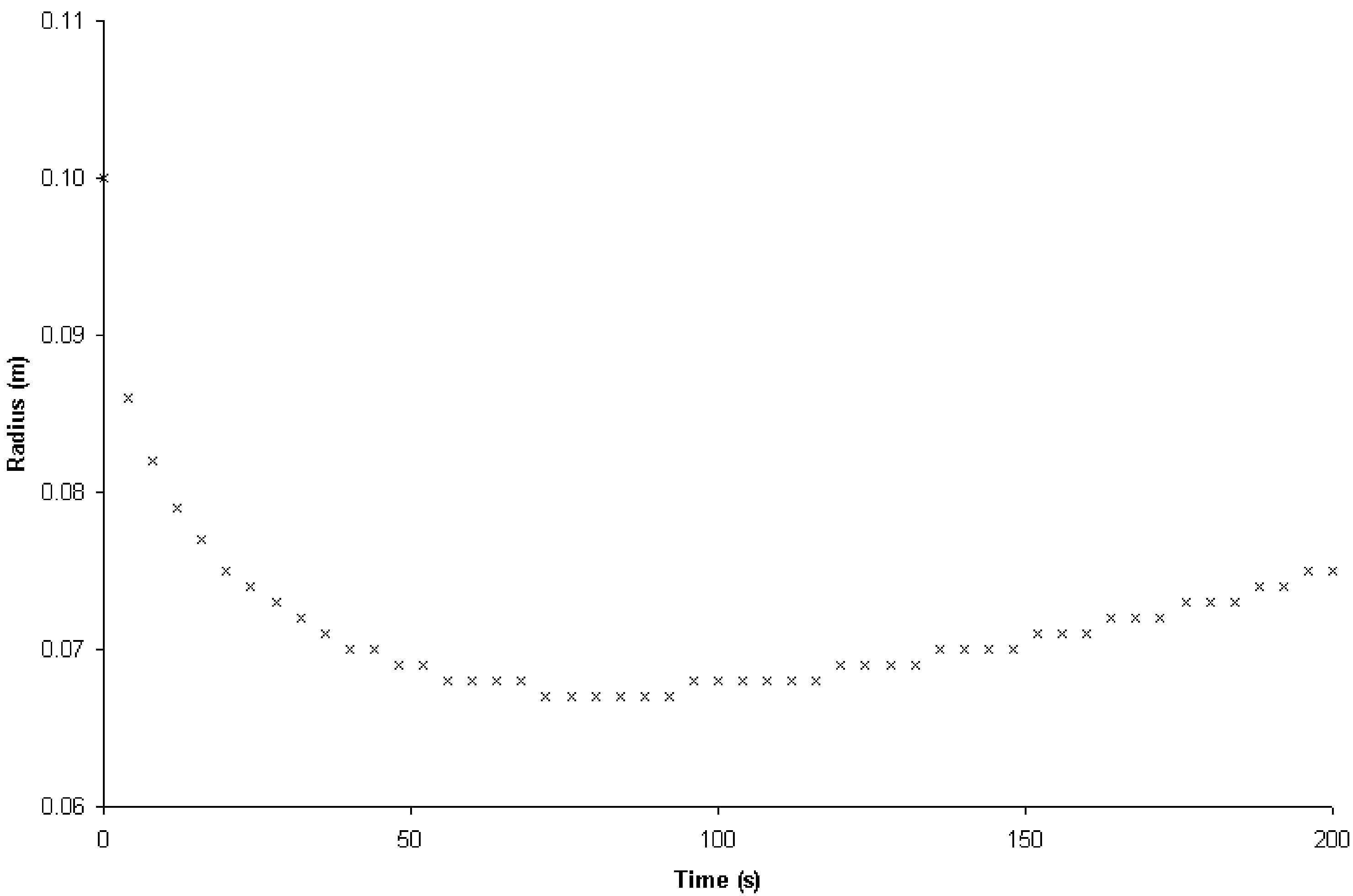


Figure 5: Time-variation of the radius of the vortex core

Analysis of fixed-radius 2D vortex decay (1)

Navier-Stokes equations for 2D axisymmetric vortex flow (such that $u_r = 0$, $\partial u_\phi / \partial \phi = 0$ and $u_z = 0$):

$$\frac{\rho u_\phi^2}{r} = \frac{\partial p}{\partial r}$$

$$\frac{\partial u_\phi}{\partial t} = \nu\left(\frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r^2}\right) \qquad (1)$$

Batchelor (1967, p201) remarks that (1) "is essentially a relation for the rate of increase of angular momentum of a cylindrical shell of fluid under the action of couples exerted by friction at its inner and outer faces". The tangential stress on an element of the cylinder surface is given by

$$\sigma_{r\phi} = \rho\nu\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right)$$

By equating the rate of change of angular momentum of the fluid in a cylindrical shell, per unit length and per unit thickness, to the couple exerted by this tangential stress, we obtain

$$\frac{\partial}{\partial t}\left(2\pi\rho r^2 u_\phi\right) = \frac{\partial}{\partial r}\left\{2\pi\rho\nu r^2\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right)\right\} \qquad (2)$$

As Batchelor points out, this expression (which is the same as his (4.5.3)) leads directly to (1), establishing the connection with the Navier-Stokes equation.

Analysis of fixed-radius 2D vortex decay (2)

At this point Batchelor's analysis continues with the usual derivation of the properties of the two flow regimes of the Rankine vortex. In our case we interpret the Rankine vortex approximation as the conditions

$r \leq R$: $\quad u_\phi = r\omega$ (with ω independent of r)

$r = R$: $$\frac{\partial u_\phi}{\partial r} = 0$$

where R is the radius of the rigid core, identified here as the radius at which the swirl (azimuthal) velocity is at a maximum. Additionally, in order to obtain an analytic solution, let us suppose that R is unchanging in time. Integrating (2) with these conditions, we obtain

$$\frac{\partial}{\partial t}\left(\omega \int_0^R r^3\,dr\right) = \left[\nu r^2\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right)\right]_0^R$$

$$\frac{\partial \omega}{\partial t} = -\frac{4\nu\omega}{R^2} \qquad (3)$$

Integrate again: $\quad \omega(t) = \omega(0)\exp\left(-4\nu t/R^2\right)$

$$u_\phi(r,t) = u_\phi(r,0)\exp\left(-4\nu t/R^2\right) \text{ for } r \leq R \qquad (4)$$

Figure 6: Decay of maximum swirl velocity

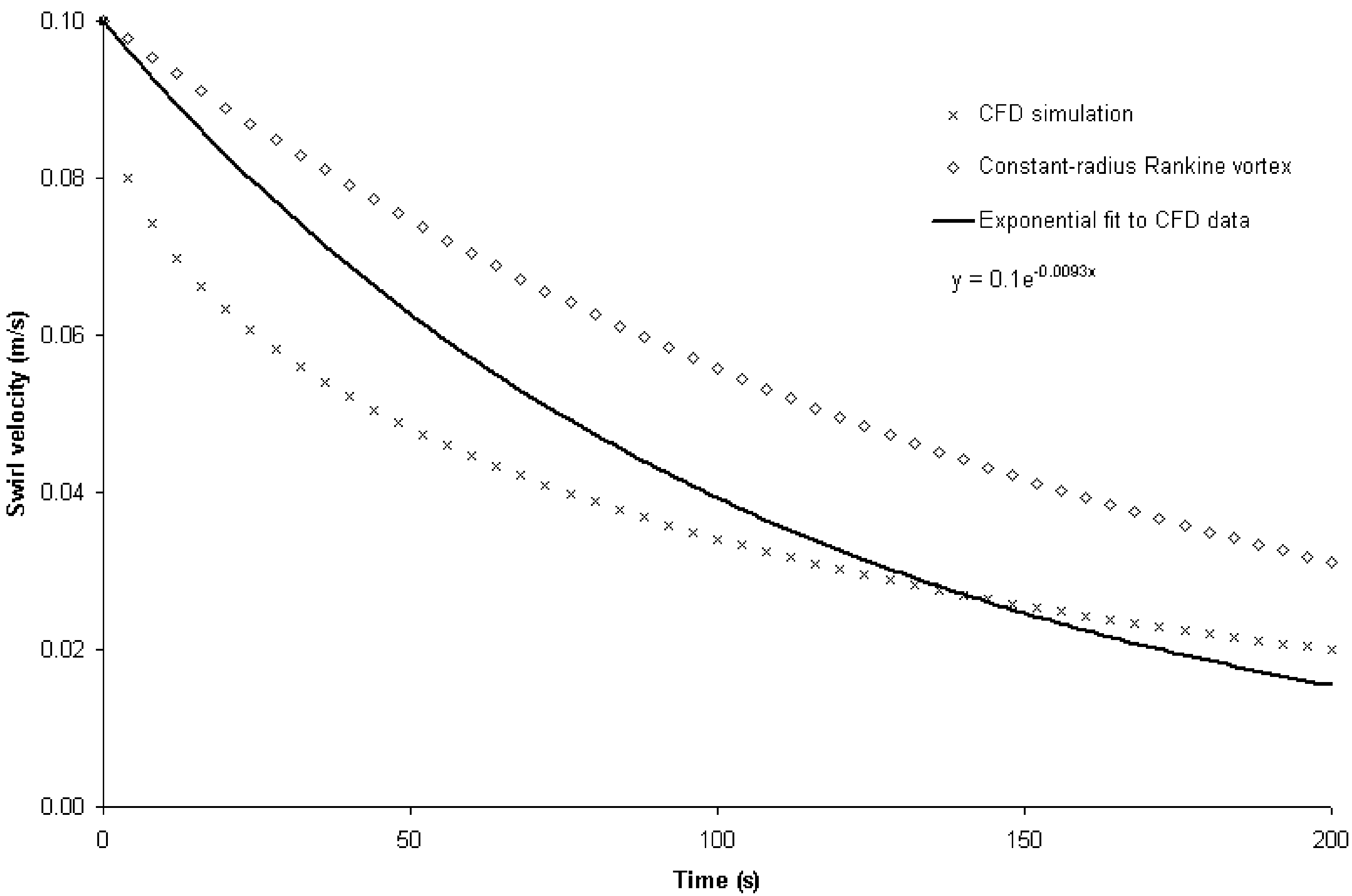
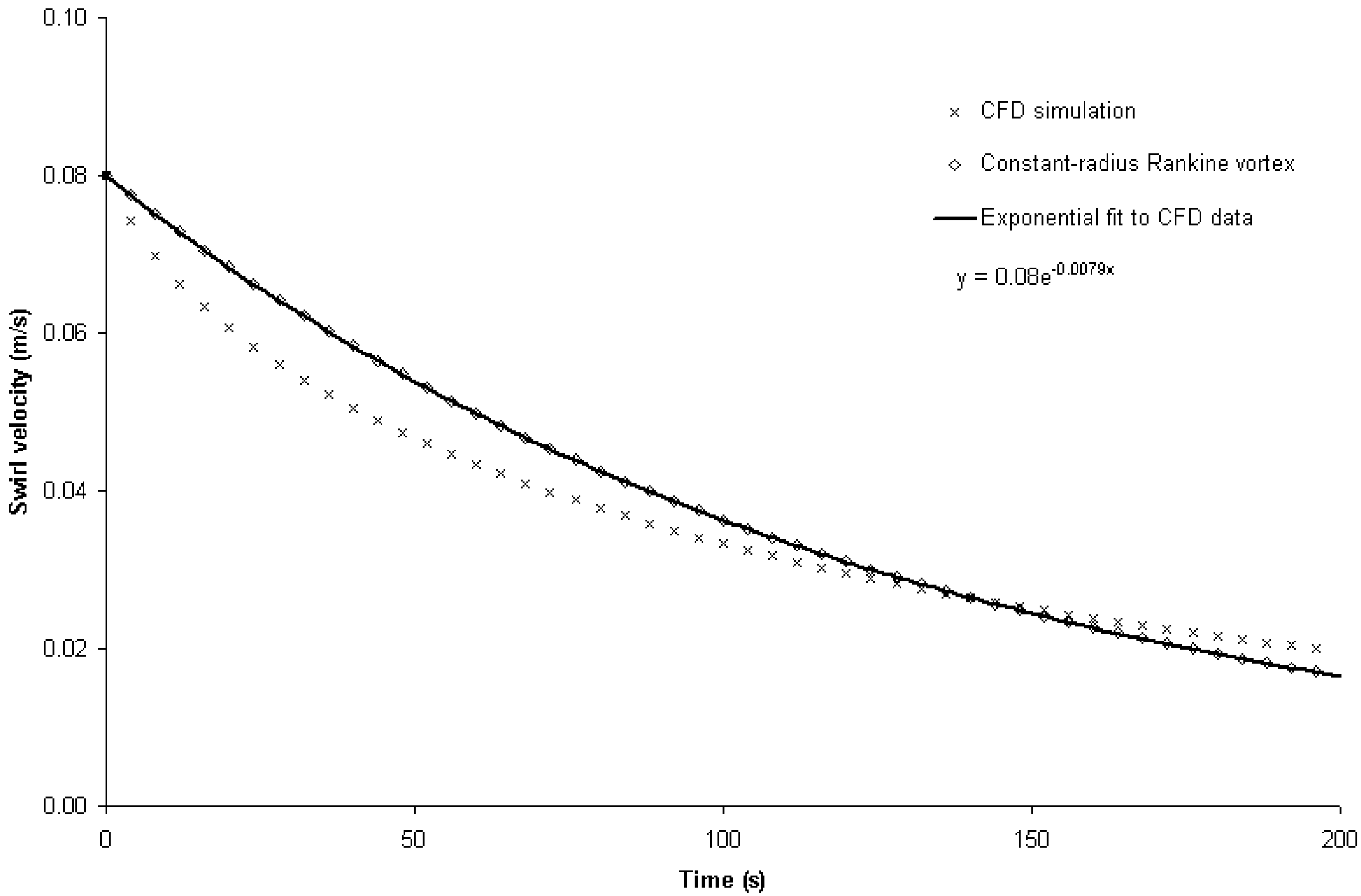


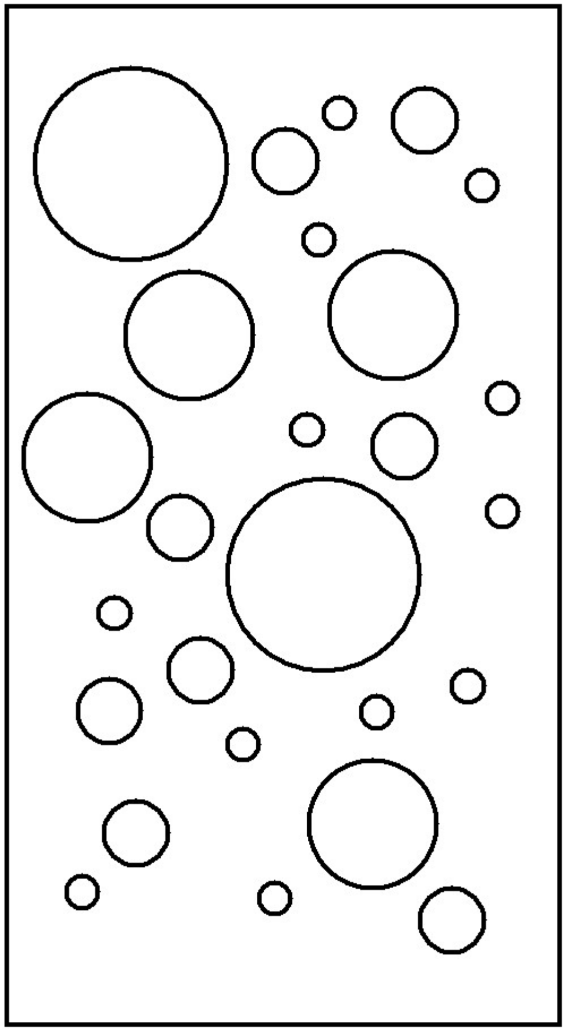
Figure 7: Decay with shifted time origin



An ensemble of vortices (1)

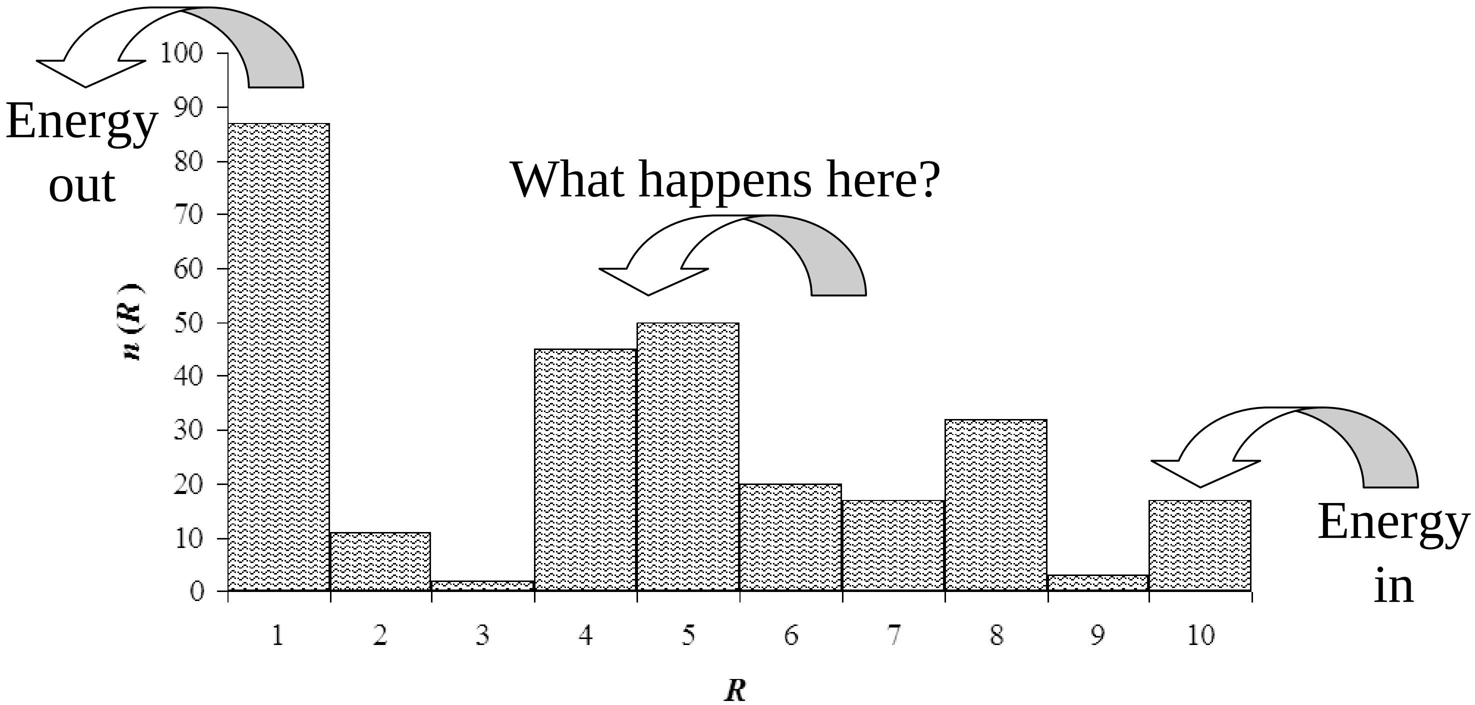
- Suppose that we have a large number of different-sized fixed-radius 2D Rankine vortices in a fluid
  - Without a continual injection of energy, the motion will decay
  - Do work on the system, *i.e*. inject ordered energy in the form of large ('integral scale') vortices
  - Extract heat from the system, *i.e*. eliminate disordered energy in the form of small ('dissipation scale') vortices
  - Continue until the general flow pattern appears to have reached a state of 'dynamic equilibrium'
- What is the equilibrium distribution of vortices?

An ensemble of vortices (2)



"The interaction of two eddies can be decomposed into (a) the convection of one by the other and (b) the shearing of one by the other... the first of these effects results only in a phase change of the associated Fourier coefficient and is not dynamically significant. The second results in the internal distortion of the eddies with the transfer of energy to a smaller size of disturbance. If the interacting eddies differ greatly in size, then it is physically plausible to argue that the dynamically irrelevant phase change is the main effect. From there it is only a step to argue that we can assume that the non-linear coupling of modes is to some degree local in wavenumber space." (David McComb)

An ensemble of vortices (3)



- McComb => A vortex will change size (*i.e.* shift from one size bin to another) only if it hits another vortex of similar size (*i.e.* within the same size bin)
  - Prob. of evacuation of a size bin ∝ total volume of vortices in bin, *i.e.* $\propto n(R)R^2$
  - This holds if $n(R)$ is statistically independent of $n(Z)$ and $n(\omega)$
- In dynamic equilibrium all size bins have the same probability of evacuation
- => Equilibrium distribution of vortices $\propto 1/R^2$, *i.e.* vortices are 'space-filling'

Energy transfer

- What is the inertial (conservative) energy flux for a turbulent fluid?
  - The energy flux ε is the rate of transfer of energy per unit mass
  - Generally, ε is some function of scale, *i.e*. for vortices $\varepsilon = \varepsilon(R)$
  - But we have found that when a vortex-bearing (*i.e*. turbulent) flow is in dynamic equilibrium the vortices at all scales are space-filling
  - Thus in this case ε is *scale-independent* and it may be determined at any convenient scale… *e.g*. the smallest scale, where the lifetime of a vortex is of order of its turnover time:

$$\tau \sim \frac{R}{u_\Phi} \qquad E \sim u_\Phi^2 \qquad \varepsilon \sim -\frac{E}{\tau} \sim -\frac{u_\Phi^3}{R}$$

Energy dissipation

- Dissipative (*i.e.* non-conservative) loss of energy by vortices of radius R is determined by their viscous decay

Equation (3): $$\frac{\partial \omega}{\partial t} = -\frac{4\nu\omega}{R^2}$$

$$\varepsilon = \frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\left(\tfrac{1}{4} R^2 \omega^2\right) = -2\nu\omega^2$$

The Kolmogorov dissipation scale

- Dynamic equilibrium => At every scale $\varepsilon_{in} = \varepsilon_{out}$
  - For most scales dissipation is negligible and ε_{in} and ε_{out} are both given by the inertial (energy transfer) expressions
  - For the smallest scale – the 'dissipation' scale η – ε_{in} is given by the inertial expression but ε_{out} is given by the dissipation expression:

$$\varepsilon \sim -\frac{u_\eta^3}{\eta} \qquad \varepsilon = -2\nu\omega^2 = -2\nu\left(\frac{u_\eta}{\eta}\right)^2$$

$$\eta \sim \left(\nu^3/\varepsilon\right)^{1/4} \qquad u_\eta \sim \left(\nu\varepsilon\right)^{1/4}$$

The Kolmogorov-Obukhov –5/3 law

It has been shown that $n(R) \propto 1/R^2$. It follows that the distribution of any function scaling as $1/R$ will be uniform across the range. Specifically, the population distribution $n(k)$ of wavenumber $k \sim 1/R$ is obtained by a change of abscissa:

$$n(k) = \frac{dR}{dk} n(R) \propto -\frac{1}{k^2} \frac{1}{(1/k)^2} = \text{constant}$$

We seek an expression for the three-dimensional vortex wavenumber energy spectrum $E(k)$, *i.e.* the energy of vortices corresponding to interval k to $k + dk$. $E(k)$ can be expected to be a linear function of $n(k)$, but since this is uniform across the range $E(k)$ can be determined directly in terms of the energy of a single vortex:

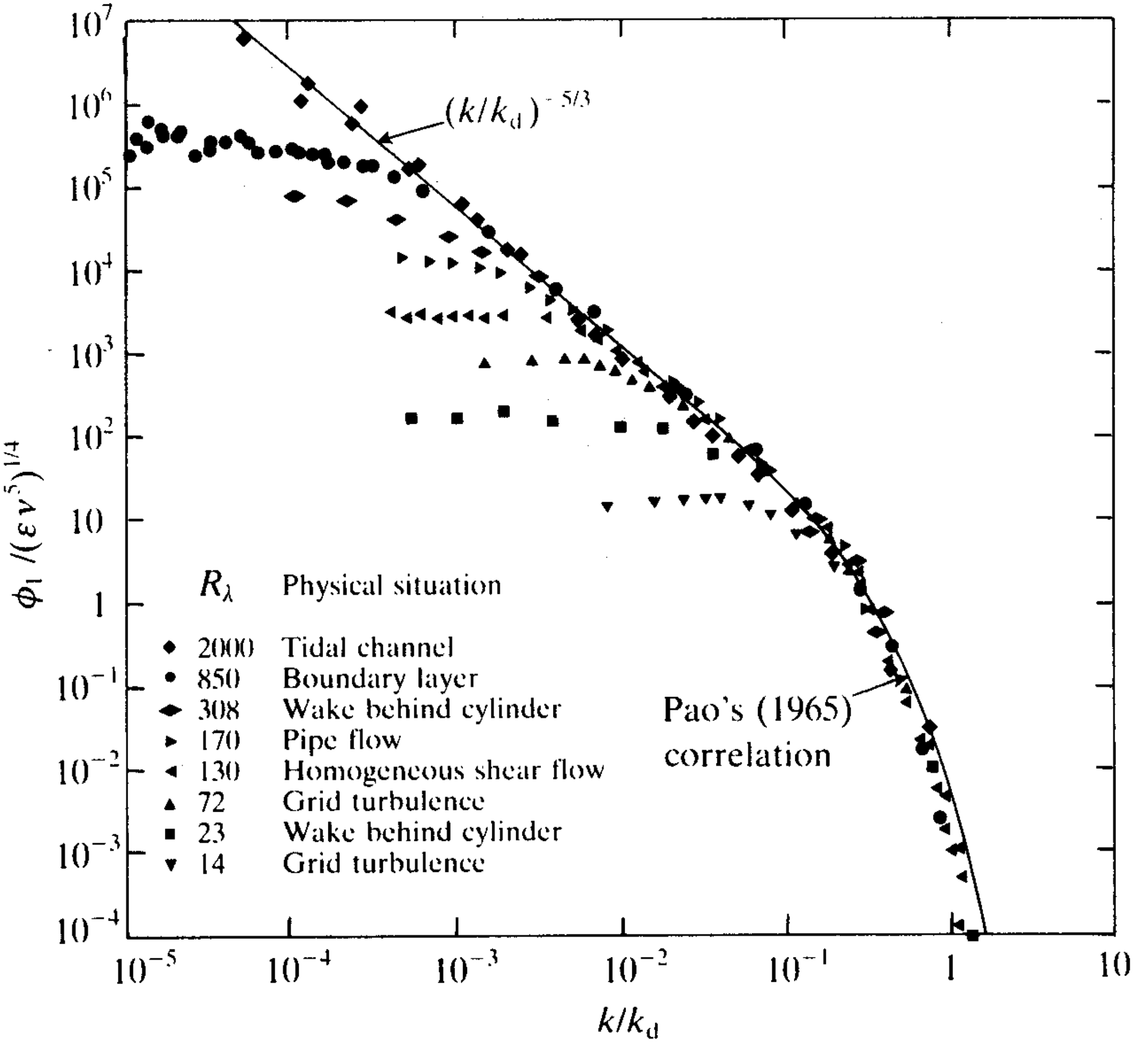
$$E(k) = \frac{\partial E}{\partial k} = \frac{\partial}{\partial k}\left(\tfrac{1}{4} R^2 \omega^2\right)$$

$$E(k) \sim \frac{\partial}{\partial k}\left(\omega^2 / k^2\right) \sim \frac{\omega^2}{k^3} \sim \frac{u_\Phi^2}{k}$$

But $\varepsilon \sim u_\Phi^3 / R$, *i.e.* $u_\Phi^3 \sim \varepsilon/k$, and substituting u_Φ we obtain the Kolmogorov-Obukhov –5/3 scaling law for isotropic turbulence:

$$E(k) = \alpha \varepsilon^{2/3} k^{-5/3}$$

The Kolmogorov-Obukhov –5/3 law



W D McComb, *The Physics of Fluid Turbulence*, OUP (Oxford, 1990)

Pao high-wavenumber correction

Pao (1965) includes an exponential correction factor in the –5/3 scaling law to take account of energy dissipation:

$$E(k) = \alpha\varepsilon^{2/3} k^{-5/3} \exp\left(-\tfrac{3}{2}\alpha (k/k_d)^{4/3}\right)$$

A similar expression is obtained if we substitute $u_\Phi(0)$ for $u_\Phi(t)$ in $E(k) \sim u_\Phi^2/k$, where $u_\Phi(0) = u_\phi(R,0)$ *etc.*

Equation (4): $$u_\phi(r,t) = u_\phi(r,0)\exp\left(-4\nu t/R^2\right) \text{ for } r \le R$$

Energy transfer: $\varepsilon \sim u_\Phi^3/R$ so that $u_\Phi \sim (\varepsilon R)^{1/3}$ and $\tau \sim R/u_\Phi \sim \varepsilon^{-1/3} R^{2/3}$

Richardson (1926) '4/3 law': $$\tau/R^2 \sim \varepsilon^{-1/3} R^{-4/3}$$

Dissipation scale: $\eta \sim \left(\nu^3/\varepsilon\right)^{1/4}$ so that $\nu\varepsilon^{-1/3} \sim \eta^{4/3}$ and $\nu\tau/R^2 \sim (\eta/R)^{4/3} \sim (k/k_d)^{4/3}$

We obtain: $$u_\Phi(t) = u_\Phi(0)\exp\left(-4\beta (k/k_d)^{4/3}\right)$$

$$E(k) = \alpha\varepsilon^{2/3} k^{-5/3} \exp\left(-8\beta (k/k_d)^{4/3}\right)$$

where β is a constant factor of order unity.

Summary

- Used CFD and analytic tools to obtain an expression for the rate of decay of a fixed-radius Rankine vortex
- Analysed the behaviour of an ensemble of such vortices
- Thereby derived key expressions in the statistical theory of turbulence:
  - The Kolmogorov dissipation scale
  - The Kolmogorov-Obukhov –5/3 law
  - The Pao high-wavenumber correction to the –5/3 law
- Thereby rehabilitating the Rankine vortex model as more than just an unphysical engineering approximation
- Begun to address the ‘unsolved problem’ of turbulence
- Experiments? Applications?

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